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ITERATED SOUSLIN FORCING, THE PRINCIPLES $\diamond(E)$ AND A GENERALISATION OF THE AXIOM SAD

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ABSTRACT

The axiom SAD was introduced in our paper with Avraham and Shelah [1]. It is a Martin's Axiom type of principle, having some of the consequences of MA plus $2^{\aleph_0} > \aleph_1$, but nonetheless provably consistent with GCH. In [1] it was shown to be consistent (with GCH) and used to demonstrate the consistency with CH of some known consequences of MA $+ 2^{\aleph_0} > \aleph_1$. In particular, SAD implies the negation of Jensen's $\diamond$ principle. In this paper we present a generalisation of SAD, let us call it SAD(E), where E will be an arbitrary stationary subset of ω_1, and show that although SAD(E) implies the negation of $\diamond(E)$, it is consistent with $\diamond$. SAD(E) resembles the axiom SA of Shelah, described in our survey article [2], and indeed is a sort of blending of the two principles SA and SAD. (In particular, Shelah proved that SA is consistent with $\diamond$ but implies the failure of some $\diamond(E)$.) Our proof (of the consistency of SAD(E) with $\diamond$) will be of interest to forcing enthusiasts, since it shows that iterated Souslin forcing *can* distinguish between different stationary sets (it was previously thought that this was not the case), and can indeed be used to establish the non-provability of the principles $\diamond(E)$ from $\diamond$ alone.

§1. Introduction

We commence by describing an axiom. We need some of the definitions from our joint paper [1]. (But be careful if you have [1] available: we change some of the definitions slightly.)

A *tree* is a poset $\boldsymbol{T} = \langle T, \leqq_T \rangle$ such that for every x in T, the set $\hat{x} = \{y \in T \mid y <_T x\}$ is well-ordered by $<_T$. The order-type of $\hat{x}$ is the height of x in $\boldsymbol{T}$, ht(x). The α'th *level* of $\boldsymbol{T}$ is the set $T_\alpha = \{x \in T \mid \text{ht}(x) = \alpha\}$. We set $T \restriction \alpha = \bigcup_{\beta<\alpha} T_\beta$. The *height* of $\boldsymbol{T}$ is the least λ such that $T_\lambda = \emptyset$, and is denoted by ht$(\boldsymbol{T})$. A *branch* of $\boldsymbol{T}$ is a maximal, totally ordered subset of $\boldsymbol{T}$; if α is its

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order-type it is an α-*branch*. An *antichain* of $\boldsymbol{T}$ is a pairwise incomparable subset of $\boldsymbol{T}$.

A tree $\boldsymbol{T}$ is *normal* iff:

(i) $\mathrm{ht}(\boldsymbol{T})$ is a limit ordinal;

(ii) if $\alpha < \beta < \mathrm{ht}(\boldsymbol{T})$ and $x \in T_\alpha$, there is a $y \in T_\beta$, $x <_T y$;

(iii) if $\alpha < \mathrm{ht}(\boldsymbol{T})$ is a limit ordinal, and if $x, y \in T_\alpha$, then $\hat{x} = \hat{y}$ implies $x = y$.

A *subtree* of a tree $\boldsymbol{T}$ is an initial section of $\boldsymbol{T}$ with the induced ordering. If $\boldsymbol{T}$ is normal, a *normal subtree* of $\boldsymbol{T}$ is a subtree of $\boldsymbol{T}$ which is a normal tree of the same height as $\boldsymbol{T}$. A *full subtree* of a normal tree $\boldsymbol{T}$ is a normal subtree, $\boldsymbol{S}$, of $\boldsymbol{T}$ such that $x \in S_\alpha$ and $x <_T y \in T_{\alpha+1}$ implies $y \in \boldsymbol{S}$.

As usual, Ω denotes the set of all non-zero countable limit ordinals.

Let $E \subseteq \Omega$. A tree $\boldsymbol{T}$ is E-*complete* iff, whenever $\delta \in E$, $\delta < \mathrm{ht}(\boldsymbol{T})$, every δ-branch of $\boldsymbol{T} \restriction \delta$ has an extension in T_δ.

Let $E \subseteq \Omega$. An *array of filters on* E is a collection $D = \{D_{\alpha,f} \mid \alpha \in E \ \&\ f \in \omega^{\underline{\alpha}}\}$ such that $D_{\alpha,f}$ is a countably complete filter on ω^α.

A *function tree* is a normal tree, $\boldsymbol{T}$, of height ω_1 such that T_α consists of elements of ω^α and the ordering of $\boldsymbol{T}$ is inclusion. A function tree $\boldsymbol{T}$ is *appropriate* for the array filters $D = \{D_{\alpha,f} \mid \alpha \in E \ \&\ f \in \omega^{\underline{\alpha}}\}$ on E iff:

(i) $\boldsymbol{T}$ is $(\Omega - E)$-complete;

(ii) if $\alpha \in E$ and $f \in T \restriction \alpha$, then there is a set $A \in D_{\alpha,f}$ such that whenever $h \in A$ is such that $f \subseteq h$ and $(\forall \xi < \alpha)(h \restriction \xi \in T)$, then $h \in T$;

(iii) if $\alpha \in E$ and $W \subseteq T \restriction \alpha$ is a full subtree of $T \restriction \alpha$, then for any $f \in W$ and any set $A \in D_{\alpha,f}$ there is $h \in A$ such that $f \subseteq h$ and $(\forall \xi < \alpha)(h \restriction \xi \in W)$.

Let SAD(E) denote the conjunction of the following statements:

(i) E is a stationary subset of Ω;

(ii) E is constructible;

(iii) GCH;

(iv) Every constructible cardinal is a cardinal;

(v) For every cardinal κ, $\mathrm{cf}(\kappa) = \mathrm{cf}^L(\kappa)$;

(vi) Every countable sequence of ordinals is constructible;

(vii) If D is a constructible array of filters on E, then every function tree which is appropriate for D has an ω_1-branch.

The axiom SAD of [1] is just SAD(Ω). Hence SAD clearly implies SAD(E) for every constructible, stationary set $E \subseteq \Omega$. The main interest in SAD(E) lies in the fact that one may simultaneously have SAD(E) and $\diamondsuit(\Omega - E)$ (although SAD(E) implies $\neg \diamondsuit(E)$). We shall prove this by an iterated forcing argument of the Jensen type described in [3], but being careful to distinguish the stationary sets E and $(\Omega - E)$. (If E contains a club set, then SAD follows easily from

SAD(E), so the only new part involves the case where $(\Omega - E)$ is also stationary.)

Throughout the paper we work in Zermelo–Fraenkel set theory, inclusive of the Axiom of Choice, and denote this theory by ZFC. Our notation is standard. Ordinals are the von Neumann ordinals, an ordinal being the set of its predecessors, and cardinals are initial ordinals. We use $\alpha, \beta, \gamma, \cdots$ to denote ordinals. The initial ordinals are denoted thus: $\omega, \omega_1, \omega_2, \cdots$; and $\aleph_\alpha$ denotes ω_α regarded as a cardinal. α^β denotes $\{f \mid f : \beta \to \alpha\}$, and we set $\alpha^{\underset{\smile}{\gamma}} = \bigcup_{\beta < \gamma} \alpha^\beta$.

The reader will find the paper hard going without a copy of our book [3] to hand. §2 presupposes an available copy of our paper [1]. A considerable acquaintance with forcing is assumed, as in, for example, Jech's book [4]. If $\boldsymbol{T}$ is a tree, BA($\boldsymbol{T}$) denotes the complete boolean algebra of all regular open subsets of the poset $\langle T, \geqq_T \rangle$ (with the usual topology), isomorphed so that T is a dense subset of BA($\boldsymbol{T}$). It should be borne in mind that the boolean ordering in BA($\boldsymbol{T}$) goes the opposite way from that of $\boldsymbol{T}$.

§2. Applications of SAD(E)

The main purpose of this paper is to show that the non-probability of $\diamondsuit(E)$ from $\diamondsuit$ can be established by a Jensen style argument as in [3], and that an SAD-type axiom can be obtained which is fairly close to the axiom SA. (Readers familar with SA will have observed that SAD(E) is very similar indeed to SA(E) as defined in our article [2].) Accordingly, we show first that SAD(E) implies the negation of $\diamondsuit(E)$.

2.1 Theorem. SAD(E) $\to \neg \diamondsuit(E)$.

Proof. Consider the following formulation of $\diamondsuit(E)$: there is a sequence $\langle f_\alpha \mid \alpha \in E \rangle$, such that $f_\alpha \in \omega^\alpha$ and whenever $f \in \omega^{\omega_1}$, *the set* $\{\alpha \in E \mid f \upharpoonright \alpha = f_\alpha\}$ is stationary.

Assuming SAD(E), we prove the above version of $\diamondsuit(E)$ fails. Let $\langle f_\alpha \mid \alpha \in E \rangle$ be given, $f_\alpha \in \omega^\alpha$. We find an $f \in \omega^{\omega_1}$ such that $f \upharpoonright \alpha \neq f_\alpha$ for *all* $\alpha \in E$.

Let $T = \{f \in \omega^{\underset{\smile}{\omega_1}} \mid (\forall \alpha \in E \cap [\mathrm{dom}(f) + 1])(f \upharpoonright \alpha \neq f_\alpha)\}$. It is easily verified that $\boldsymbol{T}$ is a function tree. (A simple diagonalisation argument shows that every member of T has arbitrarily high extensions in T.)

If T has an ω_1-branch, B, we are done, since then if $f = \bigcup B$, we have $f \in \omega^{\omega_1}$ and $(\forall \alpha \in E)(f \upharpoonright \alpha \neq f_\alpha)$. So by SAD($E$) it suffices to show that T is appropriate for some constructible array of filters on E.

We now place ourselves in L. We define an array of filters on E. Let $\alpha \in E$, $f \in \omega^{\underset{\smile}{\alpha}}$. For each $h \in \omega^\alpha$, let $A_h = \{g \in \omega^\alpha \mid g \neq h\}$. The collection $\{A_h \mid h \in \omega^\alpha\}$

clearly has the countable intersection property (if $h_n \in \omega^\alpha$, $n = 1,2,3,\cdots$, then $\bigcap_{n=1}^{\infty} A_{h_n} \neq \emptyset$). So let $D_{\alpha,f}$ be the filter generated by all countable intersections of sets A_h, $h \in \omega^\alpha$, $D_{\alpha,f}$ is a countably complete filter. This defines the array $D = \{D_{\alpha,f} \mid \alpha \in E, f \in \omega^{\mathscr{E}}\}$.

We return now to the real world, and show that T is appropriate for D. Notice that by condition (vi), T is a subset of L.

(i) Clearly, T is $(\Omega - E)$-complete.

(ii) Let $\alpha \in E$, $f \in T \restriction \alpha$. Now, the function f_α is in L. Hence $A_{f_\alpha} \in D_{\alpha,f}$. Clearly, if $h \in A_{f_\alpha}$ and $f \subseteq h$ and $(\forall \xi < \alpha)(h \restriction \xi \in T)$, then $h \in T$.

(iii) Let $\alpha \in E$, $W \subseteq T \restriction \alpha$ a full subtree of $T \restriction \alpha$, $f \in W$, $A \in D_{\alpha,f}$. Pick functions $h_n \in \omega^\alpha$, $n = 1,2,3,\cdots$, so that $\bigcap_{n=1}^{\infty} A_{h_n} \subseteq A$. Since W is closed under immediate successors (of which there are infinitely many in T at each point), it is easy to construct, by induction, a strictly increasing sequence $\langle g_n \mid n < \omega \rangle$ such that $g_n \in W$, $\sup_{n<\omega} \operatorname{dom}(g_n) = \alpha$, and $g_n \neq h_n$, with $f \subseteq g_0$. Setting $h = \bigcup_{n<\omega} g_n$, we clearly have $h \in \bigcap_{n=1}^{\infty} A_{h_n} \subseteq A$, $f \subseteq h$, $(\forall \xi < \alpha)(h \restriction \xi \in W)$.

The proof is complete. □

The above argument is just a relativisation to E of the proof that SAD implies $\neg \diamondsuit$. In a similar manner one can relativise the applications of SAD given in [1]. In particular, SAD(E) implies that every ω-colouring of every ladder system on E has a weak uniformisation. (See [1] for the relevant definitions.) This highlights the similarity between SAD(E) and SA(E), for SA(E) implies that every ω-colouring of every ladder system on E has a uniformisation. In these applications, the interest is once again in the fact that the result holds in the presence of $\diamondsuit$, and can be established by Souslin forcing.

§3. Rich sequences and Souslin forcing

In this section we describe how to make one rather special instance of SAD(E) true whilst preserving $\diamondsuit$. In the ensuing section we show how this process may be iterated in order to achieve our desired result.

We assume the reader has a copy of [3] to hand. (Our account is structured so as to be *readable* alone, but we refer to [3] for some of the proofs and individual verifications.) The following definitions are described more fully in [3].

A *Souslin tree* is a normal tree, $\boldsymbol{T}$, of height ω_1, such that:

(i) $|T_0| = 1$;

(ii) if $x \in T_\alpha$, there are $y, z \in T_{\alpha+1}$, $y \neq z$, $x <_T y$, $x <_T z$;

(iii) every antichain of $\boldsymbol{T}$ is countable.

A *Souslin algebra* is a complete, (ω, ∞)-distributive boolean algebra of cardinality $\aleph_1$ which satisfies the c.c.c. $\mathbf{B}$ will be a Souslin algebra iff there is a dense set $T \subseteq \mathbf{B}$, $\mathbf{0} \notin T$, such that $\langle T, \geqslant \mathbf{B} \rangle$ is a Souslin tree. Any such T is called a *Souslinisation* of $\mathbf{B}$. Any Souslin algebra has essentially only one Souslinisation, in the sense that if T, T' are Souslinisations of $\mathbf{B}$, then for some club set $C \subseteq \omega_1$, $\alpha \in C \rightarrow T_\alpha = T'_\alpha$.

Let $\mathbf{B}$, $\mathbf{B}'$ be Souslin algebras, $\mathbf{B}$ a complete subalgebra of $\mathbf{B}'$. The *canonical projection* $h : \mathbf{B}' \rightarrow \mathbf{B}$ is defined by $h(b') = \Lambda_{\mathbf{B}}\{b \in \mathbf{B} \mid b' \leqslant_{\mathbf{B}'} b\}$. We say that $\mathbf{B}$ is a *nice subalgebra* of $\mathbf{B}'$ iff there are Souslinisations T, T' of $\mathbf{B}$, $\mathbf{B}'$, respectively, such that for some club set $C \subseteq \omega_1$, $\alpha \in C \rightarrow T_\alpha = h[T'_\alpha]$. The "uniqueness" of Souslinisations mentioned above tells us that if such T, T' can be found, the result holds for *any* choice of T, T'.

A sequence $\langle N_\alpha \mid \alpha \in E \rangle$ is said to be *rich* iff:

(i) N_α is a countable transitive model of ZFC^-;

(ii) $\alpha \in N_\alpha$ and $N_\alpha \models$ "α is countable".

(iii) if $X \subseteq H_{\omega_1}$ is such that $\alpha < \omega_1 \rightarrow |X \cap V_\alpha| \leqslant \aleph_0$, then the set $\{\alpha \in E \mid X \cap V_\alpha \in N_\alpha\}$ is stationary in ω_1.

The existence of a rich sequence is easily seen to be equivalent to the combinatorial principle $\diamondsuit(E)$.

We fix now a stationary, co-stationary set $E \subseteq \Omega$. We assume both $\diamondsuit(E)$ and $\diamondsuit(\Omega - E)$. We fix rich sequences $\langle N_\alpha \mid \alpha \in E \rangle$ and $\langle N_\alpha \mid \alpha \in \Omega - E \rangle$. We also fix a sequence $\langle S_\alpha \mid \alpha \in E \rangle$ such that $S_\alpha \subseteq H_{\omega_1} \cap V_\alpha$ and whenever $X \subseteq H_{\omega_1}$ is such that $\alpha < \omega_1 \rightarrow |X \cap V_\alpha| \leqslant \aleph_0$, the set $\{\alpha \in E \mid X \cap V_\alpha = S_\alpha\}$ is stationary.

The following lemma (and its corollary) is standard, and easy to prove.

3.1 LEMMA. *Let $\mathbf{B}$ be a Souslin algebra, or more generally any complete Boolean algebra which satisfies the c.c.c. Let $\overset{0}{C} \in V^{(\mathbf{B})}$ be such that*

$$\| \overset{0}{C} \textit{ is a club subset of } \check{\omega}_1 \|^{\mathbf{B}} = \mathbf{1}.$$

Then there is a club set $K \subseteq \omega_1$ such that

$$\| \check{K} \subseteq \overset{0}{C} \|^{\mathbf{B}} = \mathbf{1}.$$

3.2 COROLLARY. *Let $\mathbf{B}$ be as above. If $S \subseteq \omega_1$ is stationary, then*

$$\| \check{S} \textit{ is stationary} \|^{\mathbf{B}} = \mathbf{1}.$$

A Souslin tree, $\boldsymbol{T}$, will be said to be *tubby at $\alpha \in \Omega$* iff every α-branch of $\boldsymbol{T} \upharpoonright \alpha$ which lies in N_α has an extension in T_α.

The following lemma will be used to preserve $\diamondsuit(\Omega - E)$ when we force to make SAD(E) hold.

3.3 LEMMA. *Let $C \subseteq \omega_1$ be club, and let $\boldsymbol{T}$ be a Souslin tree which is tubby at every α in $(\Omega - E) \cap C$. Set $\mathbf{B} = \mathrm{BA}(\boldsymbol{T})$. Then*

$$\|\langle N_\alpha \mid \alpha \in \Omega - E\rangle^\vee \text{ is a rich sequence}\|^{\mathbf{B}} = \mathbf{1}.$$

PROOF. Suppose the lemma is false. Then by the maximum principle we can find elements $\overset{0}{X}, \overset{0}{K}$ of $V^{(\mathbf{B})}$ such that:

$$\|\overset{0}{X} \subseteq H_{\check{\omega}_1} \;\&\; (\forall \alpha < \check{\omega}_1)(|\overset{0}{X} \cap V_\alpha| \leqq \aleph_0\| = \mathbf{1};$$

$$\|\overset{0}{K} \text{ is a club subset of } \check{\omega}_1\| = \mathbf{1};$$

$$\|(\exists \alpha \in \overset{0}{K})[\alpha \in (\Omega - E)^\vee \;\&\; \overset{0}{X} \cap V_\alpha \in \check{N}_\alpha]\| < \mathbf{1}.$$

(Since $\mathbf{B}$ is (ω, ∞)-distributive, $\|H_{\omega_1} = H_{\check{\omega}_1}\| = \mathbf{1}$.)

By 3.1 we can find a club set $A \subseteq \Omega$ such that $\|\check{A} \subseteq \overset{0}{K}\| = \mathbf{1}$. Let

$$b = \|(\exists \alpha \in \check{A})[\alpha \in (\Omega - E)^\vee \;\&\; \overset{0}{X} \cap V_\alpha \in \check{N}_\alpha]\|.$$

By the above, $b < \mathbf{1}$. We obtain a contradiction by showing that $b = \mathbf{1}$. Since T is dense in $\mathbf{B}$, it suffices to show that

$$S = \{p \in T \mid p \Vdash_T (\exists \alpha \in \check{A})[\alpha \in (\Omega - E)^\vee \;\&\; \overset{0}{X} \cap V_\alpha \in \check{N}_\alpha]\}$$

is cofinal in T.

Let $p_0 \in T$ be given. Set $T' = \{p \in T \mid p_0 \leqq_T p\}$. $\boldsymbol{T}'$ is thus a Souslin tree.

Now, since $\mathbf{B}$ is (ω, ∞)-distributive, for each $p \in T'$ and each $\alpha < \omega_1$ we can find a $q \in T'$, $p <_T q$, such that for some set $x \subseteq H_{\omega_1} \cap V_\alpha$,

$$q \Vdash \text{“}\overset{0}{X} \cap V_\alpha = \check{x}\text{”}.$$

For each $p \in T'_\cdot$ pick some maximal antichain A_p of extensions of p which decide $\overset{0}{X} \cap V_{\check{\alpha}}$ in the manner. Since $\boldsymbol{T}'$ is Souslin, each A_p is countable. Hence we can define a function $\rho : \omega_1 \to \omega_1$ by

$$\rho(\alpha) = \text{the least } \beta > \alpha \text{ such that } A_p \subseteq T' \restriction \beta \text{ for all } p \in T'_\cdot.$$

Let

$$B = \{\lambda \in \Omega \mid (\forall \alpha < \lambda)(\rho(\alpha) < \lambda)\}.$$

B is clearly closed and unbounded in ω_1.

Now, if $\lambda \in B$ and $p \in T'_\lambda$, p is comparable with some member of A_q for every $q <_{T'} p$. Hence p will decide $\overset{0}{X} \cap V_{\check{\lambda}}$.

Set $D = A \cap B \cap C$, a club subset of ω_1.

Now, we may clearly assume that T' has domain ω_1. Define a relation $R \subseteq \omega_1 \times \omega_1 \times H_{\omega_1}$ by

$$R(\lambda, p, x) \leftrightarrow \lambda \in B \ \& \ p \in T'_\lambda \ \& \ p \Vdash_T \text{“}\overset{0}{X} \cap V_{\check{\lambda}} = \check{x}\text{”}.$$

Consider the structure $\mathfrak{A} = \langle H_{\omega_1}, \varepsilon, T', R\rangle$. By recursion, define a chain $\langle \mathfrak{A}_\nu \mid \nu < \omega_1\rangle$ of elementary submodels of $\mathfrak{A}$ as follows:

$\mathfrak{A}_0 < \mathfrak{A}$ is countable;

$\mathfrak{A}_{\nu+1} < \mathfrak{A}$ is countable with $\mathfrak{A}_\nu \cup \{\mathfrak{A}_\nu\} \subseteq \mathfrak{A}_{\nu+1}$;

$\mathfrak{A}_\tau = \bigcup_{\nu<\tau} \mathfrak{A}_\nu$, if $\lim(\tau)$.

Each $\mathfrak{A}_\nu$ will, of course, be transitive. Hence $\alpha_\nu = \omega_1 \cap \mathfrak{A}_\nu \in \omega_1$, and $\langle \alpha_\nu \mid \nu < \omega_1\rangle$ is a strictly increasing, continuous sequence of ordinals in ω_1.

By richness, pick an ordinal $\alpha \in (\Omega - E) \cap D$ such that $\alpha_\alpha = \alpha$, $T' \restriction \alpha = T' \cap \alpha$, and $T' \restriction \alpha$, $R \cap (\alpha \times \alpha \times V_\alpha)$, $B \cap \alpha \in N_\alpha$.

We now attempt to carry out a construction in N_α. Since α is countable, pick a strictly increasing sequence $\langle \gamma_n \mid n < \omega\rangle$ of members of B, cofinal in α. Pick $p_0 \in T_{\gamma_0}$ and $x_0 \in V_{\gamma_0}$ so that $R(\gamma_0, p_0, x_0)$. By induction, pick $p_{n+1} \in T_{\gamma_{n+1}}$, $p_n <_T p_{n+1}$, and $x_{n+1} \in V_{\gamma_{n+1}}$, $x_n \subseteq x_{n+1}$, so that $R(\gamma_{n+1}, p_{n+1}, x_{n+1})$. Set $x = \bigcup_{n<\omega} x_n$.

Returning now to the real world, notice that because $\mathfrak{A}_\alpha < \mathfrak{A}$, the above construction was indeed possible. Now, $\langle p_n \mid n < \omega\rangle$ defines an α-branch of $T' \restriction \alpha$. But the branch is in N_α. Hence as T is tubby at every member of $(\Omega - E) \cap C$, there is a $p' \in T'_\alpha$ such that p' extends each p_n. But then $p' \Vdash_T \text{“}\overset{0}{X} \cap V_{\check{\alpha}} = \check{x}\text{”}$, so as we have $x \in N_\alpha$, it follows that $p' \Vdash_T \text{“}\overset{0}{X} \cap V_{\check{\alpha}} \in \check{N}_{\check{\alpha}}\text{”}$. But $\alpha \in A \cap (\Omega - E)$, so this means that $p' \in S$, and the proof is complete. □

An array of filters $D = \{D_{\alpha,f} \mid \alpha \in E \ \& \ f \in \omega^{\alpha}\}$ on E is said to be *principal* if each filter $D_{\alpha,f}$ is principal.

The following lemma provides the means by which we shall be able to iterate in order to make SAD(E) hold.

3.4 Lemma. *Let $C \subseteq \omega_1$ be club, and let T be a Souslin tree which is tubby at every α in $(\Omega - E) \cap C$. Let $D = \{D_{\alpha,f} \mid \alpha \in E \ \& \ f \in \omega^{\alpha}\}$ be a principal array of filters. Let $\mathbf{B} = \mathrm{BA}(T)$, and let $\overset{0}{T} \in V^{\mathbf{B}}$ be such that*

$$\| \overset{0}{T} \textit{ is a function tree which is appropriate for } \check{D} \|^{\mathbf{B}} = \mathbf{1}.$$

Then there is a Souslin tree, $\tilde{T}$, such that, setting $\mathbf{B} = \mathrm{BA}(\tilde{T})$:

(i) *for some club set $\tilde{C} \subseteq C$, $\tilde{T}$ is tubby at every α in $(\Omega - E) \cap \tilde{C}$;*

(ii) *$\mathbf{B}$ is a nice subalgebra of $\tilde{\mathbf{B}}$;*

(iii) *$\| \overset{0}{T}$ has an $\check{\omega}_1$-branch $\|^{\tilde{\mathbf{B}}} = \mathbf{1}$.*

Proof. The proof is an adaptation of the proof of 3.2 in [1]. We construct $\tilde{T}$ to be recursion on the levels, simultaneously constructing a strictly increasing, continuous function $\gamma : \omega_1 \to \omega_1$. The elements of $\tilde{T}_\alpha$ will be pairs (x, f) such that $x \in T_{\gamma(\alpha)}$, $f \in \omega^\alpha$, and $x \Vdash_T$ "$\check{f} \in \overset{0}{T}$". The ordering of $\tilde{T}$ will be $(x, f) \leqq_{\tilde{T}} (x', f') \leftrightarrow x \leqq_T x'$ & $f \subseteq f'$. The construction will be carried out so as to preserve the following conditions:

(I) If $(x, f) \in \tilde{T}_\alpha$ and $\alpha < \beta$ and $x' \in T_{\gamma(\beta)}$, $x <_T x'$, then for some $f' \supseteq f$, $(x', f') \in \tilde{T}_\beta$;

(II) If $(x, f) \in \tilde{T}_\alpha$ and $x' \in T_{\gamma(\alpha+1)}$, $x <_T x'$, then for each $n \in \omega$, either $x' \Vdash_T$ "$\check{f} \cup \{(\check{n}, \check{\alpha})\} \in \overset{0}{T}$" or else $x' \Vdash_T$ "$\check{f} \cup \{(\check{n}, \check{\alpha})\} \notin \overset{0}{T}$";

(III) If $x \in T_{\gamma(\alpha)}$ and $W = \{f \mid (\exists x' \leqq_T x)[(x', f) \in \tilde{T}]\}$, then $x \Vdash_T$ "$\check{W}$ is a full subtree of $\overset{0}{T} \restriction (\check{\alpha} + 1)$".

Defining $h : \tilde{T} \to T$ by $h(x, f) = x$, h induces (see [3]) a complete embedding e of $\mathbf{B}$ into $\tilde{\mathbf{B}} = \mathrm{BA}(\tilde{T})$ for which h is the restriction of $\tilde{T}$ of the canonical projection. (This uses (I) above.) It follows at once that, up to isomorphism, $\tilde{\mathbf{B}}$ is a nice extension of $\mathbf{B}$. This will give us part (ii) of the lemma. From now on we concentrate entirely upon the construction of $\tilde{T}$ and properties (i), (iii) of the lemma.

We may assume that T has domain ω_1. To commence the construction, we set:

$$\tilde{T}_0 = \{(x, \varnothing) \mid x \in T_0\};$$

$$\gamma(0) = 0.$$

Now suppose that $\tilde{T}_\alpha$, $\gamma(\alpha)$ are defined. For each $(x, f) \in \tilde{T}_\alpha$ and each $n \in \omega$, let $A_{x,f,n}$ be a maximal pairwise incomparable set of extensions x' of x in T such that either $x' \Vdash_T$ "$\check{f} \cup \{(\check{n}, \check{\alpha})\} \in \overset{0}{T}$" or $x' \Vdash_T$ "$\check{f} \cup \{(\check{n}, \check{\alpha})\} \notin \overset{0}{T}$". Since T is Souslin, each set $A_{x,f,n}$ is countable. So we may define $\gamma(\alpha + 1)$ to be the least $\gamma > \gamma(\alpha)$ such that for all $(x, f) \in \tilde{T}_\alpha$ and all $n \in \omega$, $A_{x,f,n} \subseteq T \restriction \gamma$. For each $(x, f) \in \tilde{T}_\alpha$ and each $x' \in T_{\gamma(\alpha+1)}$ with $x <_T x'$, and for each $n \in \omega$ with $x' \Vdash_T$ "$\check{f} \cup \{(\check{n}, \check{\alpha})\} \in \overset{0}{T}$", put $(x', f \cup \{(n, \alpha)\})$ into $\tilde{T}_{\alpha+1}$ (to extend (x, f)). This definition clearly preserves conditions (I)–(III).

Now suppose that $\alpha \in E$ and we have defined $\tilde{T} \upharpoonright \alpha$, $\gamma \upharpoonright \alpha$. Set $\gamma(\alpha) = \sup_{\beta<\alpha} \gamma(\beta)$. Let $x_0 \in T_{\gamma(\alpha)}$. Set

$$W(x_0) = \{f \mid (\exists x <_T x_0)[(x, f) \in \tilde{T} \upharpoonright \alpha]\}.$$

By condition (III),

(a) $$x_0 \Vdash_T \text{“} W(x_0)^\vee \text{ is a full subtree of } \overset{0}{T} \upharpoonright \check{\alpha} \text{”}.$$

Let $(x, f) \in \tilde{T} \upharpoonright \alpha$, $x <_T x_0$. Then $x_0 \Vdash_T \text{“}\check{f} \in \overset{0}{T}\text{”}$, so as

$$\| \overset{0}{T} \text{ is appropriate for } \check{D} \|^{\mathbf{B}} = \mathbf{1},$$

we have

(b) $$x_0 \Vdash_T \text{“if } h \supseteq \check{f} \;\&\; h \in \bigcap D_{\alpha,f}^\vee \;\&\; (\forall \xi < \check{\alpha})(h \upharpoonright \xi \in \overset{0}{T}), \text{ then } h \in \overset{0}{T}\text{”}.$$

But because $\| \overset{0}{T}$ is appropriate for $\check{D} \|^{\mathbf{B}} = \mathbf{1}$, (a) gives

(c) $$x_0 \Vdash \text{“there is } h \supseteq \check{f} \text{ such that } h \in \bigcap D_{\alpha,f}^\vee \text{ and } (\forall \xi < \check{\alpha})(h \upharpoonright \xi \in W(x_0)^\vee)\text{”}.$$

Now by (c) there must be an $h \supseteq f$, $h \in \bigcap D_{\alpha,f}$, such that $(\forall \xi < \alpha)$ $(h \upharpoonright \xi \in W(x_0))$. Then by (a) and (b), $x_0 \vDash_T \text{“}\check{h} \in \overset{0}{T}\text{”}$.

Setting $U(x_0) = \{(x, f) \mid x <_T x_0 \;\&\; (x, f) \in \tilde{T} \mid \alpha\}$ now, if $U(x_0) \cap S_\alpha$ is cofinal in $U(x_0)$ (under $\leqq_{\tilde{T}}$), then to each pair $(x, f) \in U(x_0) \cap S_\alpha$ we pick an h as above and put (x_0, h) into $\tilde{T}_\alpha$. And if $U(x_0) \cap S_\alpha$ is not cofinal in $U(x_0)$, then to each pair $(x, f) \in U(x_0)$ we pick an h as above and put (x_0, h) into $\tilde{T}_\alpha$. Do this for each x_0 in $T_{\gamma(\alpha)}$. It is easily seen that this defines $\tilde{T}_\alpha$ to preserve (I)–(III).

Finally, suppose $\alpha \in \Omega - E$ and $\tilde{T} \upharpoonright \alpha$, $\gamma \upharpoonright \alpha$ are defined. As before we set $\gamma(\alpha) = \sup_{\beta<\alpha} \gamma(\beta)$. Let $x \in T_{\gamma(\alpha)}$, $(x_0, f_0) \in \tilde{T} \upharpoonright \alpha_1$, $x_0 <_T x$. Using property (I), an easy induction produces a sequence $\langle (x_n, f_n) \mid < \omega \rangle$ such that $x_0 <_T x_1 <_T x_2 <_T \cdots <_T x$ and $f_0 \subseteq f_1 \subseteq f_2 \subseteq \cdots$ and $(x_n, f_n) \in \tilde{T} \mid \alpha$. Then $x_0 \Vdash_T \text{“}\langle f_n \mid n < \omega \rangle^\vee$ determines an $\check{\alpha}$-branch of $\overset{0}{T}\text{”}$. So, as $\| \overset{0}{T}$ is $(\Omega - E)^\vee$-complete$\|^{\mathbf{B}} = \mathbf{1}$, $x_0 \Vdash \text{“}\bigcup_{n<\omega} f_n^\vee \in \overset{0}{T}\text{”}$. Let $f = \bigcup_{n<\omega} f_n$. Put (x, f) into $\tilde{T}_\alpha$. Do this for all $x \in T_{\gamma(\alpha)}$, $(x_0, f_0) \in \tilde{T} \upharpoonright \alpha$, $x_0 <_T x$. If, in addition, $a = \gamma(\alpha) \in C$, then whenever $\langle (x_\nu, f_\nu) \mid \nu < \alpha \rangle$ is an α-branch of $\tilde{T} \upharpoonright \alpha$ which lies in N_α, we let x be the unique extension of $\langle x_\nu \mid \nu < \alpha \rangle$ in $T_{\gamma(\alpha)}$ and set $f = \bigcup_{\nu<\alpha} f_\nu$ and put (x, f) into $\tilde{T}_\alpha$. Since $\| \overset{0}{T}$ is $(\Omega - E)^\vee$-complete$\|^{\mathbf{B}} = \mathbf{1}$, $x \Vdash_T \text{“}\check{f} \in \overset{0}{T}\text{”}$, so this is in order. It is easily seen that this defines $\tilde{T}_\alpha$ to preserve (I)–(III).

The definition of $\tilde{T}$ is complete. Let $\tilde{C} = C \cap \{\alpha \mid \gamma(\alpha) = \alpha\}$. It is immediate now that $\tilde{T}$ is tubby at every α in $(\Omega - E) \cap \tilde{C}$. And it is easily seen that $\| \overset{0}{T}$ has an $\check{\omega}_1$-branch$\|^{\mathbf{B}} = \mathbf{1}$. It remains to verify that $\tilde{T}$ is Souslin.

Let A be a maximal antichain in T, and set $\bar{A} = \{u \in \tilde{T} \mid (\exists a \in A)(\alpha \leqq_{\tilde{T}} u)\}$. $\bar{A}$ is cofinal in $\tilde{T}$. For each $(x, f) \in \tilde{T}$, let $E_{x,f}$ be a maximal pairwise incomparable subset of the set

$$\{x' \in T \mid x <_T x' \ \&\ (\exists f' \supseteq f)[(x', f') \in \bar{A}]\}.$$

Since T is Souslin, $E_{x,f}$ is countable. Let

$$K = \{\alpha \in \tilde{C} \mid \tilde{T} \restriction \alpha = \tilde{T} \cap V_\alpha \ \&\ (\forall (x, f) \in \tilde{T} \restriction \alpha)(E_{x,f} \subseteq T \restriction \alpha)\}.$$

K is clearly closed and unbounded in ω_1. So by $\diamondsuit(E)$ we can pick an $\alpha \in K \cap E$ such that $\bar{A} \cap \tilde{T} \restriction \alpha = S_\alpha$.

Let $x_0 \in T_\alpha$. Let $(x, f) \in U(x_0)$. Pick $f_0 \supseteq f$ with $(x_0, f_0) \in \tilde{T}_\alpha$. Since $\bar{A}$ is cofinal in $\tilde{T}$ there is an $(x', f') \in \bar{A}$ extending (x_0, f_0). Since $E_{x,f} \subseteq T \restriction \alpha$, we cannot have $x' \in E_{x,f}$. So, as (x', f') extends (x, f) it must be the case that for some $x'' <_T x_0$, $x'' \in E_{x,f}$. Pick $f'' \supseteq f$ so that $(x'', f'') \in \bar{A}$. Then (x'', f'') extends (x, f), lies in S_α, and is a member of $U(x_0)$. Thus $S_\alpha \cap U(x_0)$ is cofinal in $U(x_0)$.

Hence, by construction, every member of $\tilde{T}_\alpha$ extends a member of $\bar{A}$. Thus $A = A \cap \tilde{T} \restriction \alpha$, and A must be countable. That completes the proof of the lemma. □

§4. Proof of the main theorem

In this section we describe how to iterate the forcing described in §3 in order to obtain the consistency of SAD(E) and $\diamondsuit(\Omega - E)$. We adapt the iteration technique of Jensen described in [3]. We assume the reader still has his copy of this book to hand, and shall refer to it constantly. As before, $E \subseteq \Omega$ will be a fixed stationary, costationary set.

We assume $V = L$. This implies the existence of a sequence $\langle N_\alpha \mid \alpha \in \Omega \rangle$ such that:

(i) N_α is a countable transitive model of ZFC$^-$;

(ii) $\alpha \in N_\alpha$ and $N_\alpha \models$ "α is countable";

(iii) if $X \subseteq H_{\omega_1}$ is such that $\alpha < \omega_1 \rightarrow |X \cap V_\alpha| \leqq \aleph_0$, then there is a club set $C \subseteq \omega_1$ such that $\alpha \in C \rightarrow X \cap V_\alpha \in N_\alpha$.

Notice that $\langle N_\alpha \mid \alpha \in E \rangle$ and $\langle N_\alpha \mid \alpha \in \Omega - E \rangle$ are both rich sequences in the sense of §3. (The existence of a sequence $\langle N_\alpha \mid \alpha \in \Omega \rangle$ as above is, of course, equivalent to the combinatorial principle $\diamondsuit^*$.) By $V = L$, we also may fix a $\diamondsuit$-sequence $\langle S_\alpha \mid \alpha \in E \rangle$ of the kind described in §3. Also as a consequence of $V = L$, there is a $\square$-sequence: that is, a sequence $\langle A_\lambda \mid \lambda < \omega_2 \ \&\ \lim(\lambda) \rangle$ such that for all limit ordinals $\lambda < \omega_2$:

(i) A_λ is a closed unbounded subset of λ;

(ii) $\mathrm{cf}(\lambda) = \omega \rightarrow \mathrm{otp}(A_\lambda) < \omega_1$;

(iii) $\mathrm{cf}(\lambda) = \omega_1 \rightarrow \mathrm{opt}(A_\lambda) = \omega_1$;

(iv) if $\alpha \in A_\lambda$ and $\alpha = \sup(A_\lambda \cap \alpha)$, then $A_\alpha = A_\lambda \cap \alpha$.

We fix some bijection $\phi : \omega_2 \times \omega_2 \times \omega_2 \leftrightarrow \omega_2$ such that $\phi(\alpha, \beta, \gamma) \geqq \alpha, \beta, \gamma$ for all α, β, γ. Let i, j, k be the inverse functions to ϕ.

We define a sequence $\langle \mathbf{B}_\nu \mid \nu < \omega_2 \rangle$ of complete boolean algebras such that:

(i) $\mathbf{B}_0 = \mathbf{2}$;

(ii) $0 < \nu < \omega_2 \rightarrow \mathbf{B}_\nu$ is a Souslin algebra;

(iii) $0 < \nu < \tau < \omega_2 \rightarrow \mathbf{B}_\nu$ is a nice subalgebra of $\mathbf{B}_\tau$.

The definition is by recursion on ν. As it proceeds we define Souslinisations T^ν of the $\mathbf{B}_\nu$ and club sets C_τ, $C_{\nu\tau}$, $0 < \tau < \nu$, such that:

(iv) T^ν is tubby at every α in $(\Omega - E) \cap C_\nu$;

(v) $\alpha \in C_{\nu\tau} \rightarrow [T^\nu \restriction \alpha = T^\nu \cap V_\alpha \quad \& \quad T^\tau \restriction \alpha = T^\tau \cap V_\alpha \quad \& \quad T^\nu_\alpha = h_{\tau\nu}[T^\tau_\alpha]]$,

where $h_{\tau\nu} : \mathbf{B}_\tau \rightarrow \mathbf{B}_\nu$ is the canonical projection.

By GCH, there are $\aleph_2$ principle filter arrays. Let $\langle D_\alpha \mid \alpha < \omega_2 \rangle$ enumerate them.

For each α, we shall set

$$X_\alpha = \{ \overset{0}{T} \in V^{(\mathbf{B}_\alpha)} \mid \| \overset{0}{T} \text{ is a function tree} \|^{\mathbf{B}_\alpha} = \mathbf{1} \}.$$

We fix an enumeration $\langle \overset{0}{T}{}^{\alpha,\xi} \mid \xi < \omega_2 \rangle$ of X_α.

Suppose $\langle \mathbf{B}_\tau \mid \tau \leqq \nu \rangle$ is defined. Let $\alpha = i(\nu)$, $\beta = j(\nu)$, $\gamma = k(\nu)$. If

$$\| \overset{0}{T}{}^{\alpha,\gamma} \text{ is appropriate for } \check{D}_\beta \|^{\mathbf{B}_\nu} = 1,$$

we obtain $T^{\nu+1}$, $\mathbf{B}_{\nu+1}$, $C_{\nu+1}$ from C_ν, T^ν, D_β, $\overset{0}{T}{}^{\alpha,\gamma}$ in the same way that $\tilde{T}$, $\tilde{\mathbf{B}}$, $\tilde{C}$ were obtained from C, T, D, $\overset{0}{T}$ in 3.4. We let $C_{\nu+1,\nu}$ be any club set with $\alpha \in C_{\nu+1,\nu} \rightarrow h_{\nu+1,\nu}[T^{\nu+1}_\alpha] = T^\nu_\alpha$, and set $C_{\nu+1,\tau} = C_{\nu+1,\nu} \cap C_{\nu,\tau}$ for $0 < \tau < \nu$.

If $\| \overset{0}{T}{}^{\alpha,\gamma}$ is appropriate for $D_\beta \|^{\mathbf{B}_\nu} < \mathbf{1}$, we set $T^{\nu+1} = T^\nu$, $\mathbf{B}_{\nu+1} = \mathbf{B}_\nu$, $C_{\nu+1} = C_\nu$, $C_{\nu+1,\nu} = \Omega$, and $C_{\nu+1,\tau} = C_{\nu\tau}$ for $0 < \tau < \nu$.

Now suppose we have defined $\langle \mathbf{B}_\nu \mid \nu < \lambda \rangle$ where λ is a limit ordinal cofinal with ω. Let $\theta = \mathrm{otp}(A_\lambda)$, and let $\langle \lambda(\nu) \mid \nu < \theta \rangle$ be the canonical enumeration of A_λ. Notice that $\theta \leqq \lambda$ and $\theta < \omega_1$. We define T^λ (and $\mathbf{B}_\lambda$) from $\langle T^{\lambda(\nu)} \mid \nu < \theta \rangle$.

Set $C = \bigcap_{\nu < \tau < \theta} [C_{\lambda(\nu),\lambda(\tau)} - \theta] \cup \{0\}$. Let $\langle c_\nu \mid \nu < \omega_1 \rangle$ be the canonical enumeration of C and define $C_{\lambda,\nu} = \{\alpha \mid c_\alpha = \alpha\}$ for $0 < \nu < \lambda$.

Set $C_\lambda = (\bigcap_{\nu < \lambda} C_{\lambda(\nu)}) \cap C_{\lambda,1}$.

For all $\alpha < \omega_1$, $T^{\lambda(\nu)}_{c_\alpha} = h_{\lambda(\tau),\lambda(\nu)}[T^{\lambda(\tau)}_{c_\alpha}]$ for each $\nu < \tau < \theta$, so we may define

$$T^* = \{x = \langle x_\nu \mid \nu < \theta \rangle \mid (\exists \gamma < \omega_1)(\nu < \theta \rightarrow x_\nu \in T^{\lambda(\nu)}_{c_\alpha} \ \&$$
$$\& \ \nu < \tau < \theta \rightarrow x_\nu = h_{\lambda(\tau)\lambda(\nu)}(x_\tau))\}.$$

Partially order T^* by

$$x \leqq^* y \leftrightarrow (\forall \nu < \theta)(x_\nu \leqq_{T^{\lambda(\nu)}} y_\nu).$$

It is easily seen that $(T^*, \leqq^*)$ is a tree, with $x \in T^*_\gamma \leftrightarrow (\forall \nu < \theta)(x_\nu \in T^{\lambda(\nu)}_{c_\alpha})$.

We define $T^\lambda \subseteq T^*$ by recursion on the levels. To commence we let T^λ_0 be the minimal element of T^*. In general we shall have $T^\lambda_\gamma \subseteq T^*_{c_\alpha}$.

Suppose $T^\lambda \restriction (\alpha + 1)$ has been defined. We define $T_{\alpha+1}$ as follows. For each triple (ν, y, x) with $\nu < \theta$, $x \in T_\alpha$, $y \in T^{\lambda(\nu)}_{c_{\alpha+1}}$, $x_\nu \leqq_{T^{\lambda(\nu)}} y$, let $s = s(\nu, y, x) \in T^*_{\alpha+1}$ be such that $x \leqq^* s$ and $s_\nu = y$. (The existence of such an s is demonstrated on p. 90 of [3].) We put one such $s(\nu, y, x)$ into $T^\lambda_{\alpha+1}$ for each triple (ν, y, x). The actual choice of $s(\nu, y, x)$ is irrelevant except when $\alpha = 0$ and $c_1 \in E$ and

(*) for all $\nu < \theta$, $y \in T^{\lambda(\nu)}_{c_1}$, there is a pair $\langle \tau, z \rangle \in S_{c_1}$ such that $\tau \geqq \nu$, $\tau < \theta$, $z \in T^{\lambda(\tau)} \restriction c_1$, and $h_{\lambda(\tau)\lambda(\nu)}(z) \leqq_{T^{\lambda(\nu)}} y$.

In this case we select $s(\nu, y, x)$ so that for some $\langle \tau, z \rangle \in S_{c_1}$, $z \in T^{\lambda(\tau)} \restriction c_1$ and $z \leqq_{T^{\lambda(\tau)}} s_\tau$.

That defines $T^\lambda_{\alpha+1}$ completely, except when $\alpha = 0$ and $c_1 \in (\Omega - E) \cap (\bigcap_{\nu < \lambda} C_{\lambda(\nu)})$, when we enlarge T^λ_1 as follows. Suppose $\langle T^{\lambda(\nu)} \restriction \nu < \theta \rangle$, $\langle h_{\lambda(\tau)\lambda(\nu)} \restriction (T^{\lambda(\tau)} \restriction c_1) \mid \nu < \theta \rangle \in N_{c_1}$, that $\langle \gamma(\nu) \mid \nu < \theta \rangle$ is a strictly increasing sequence of ordinals cofinal in c_1, which lies in N_{c_1}, and that $\langle x_\nu \mid \nu < \theta \rangle$ is a sequence with $x_\nu \in T^{\lambda(\nu)}_{\gamma(\nu)}$ and $\nu < \tau < \theta \rightarrow x_\nu \leqq_{T^{\lambda(\nu)}} h_{\lambda(\tau),\lambda(\nu)}(x_\tau)$, also lying in N_{c_1}. Suppose further that for each $\nu < \theta$, $\langle h_{\lambda(\tau),\lambda(\nu)}(x_\tau) \mid \nu < \tau < \theta \rangle$ defines a c_1-branch of $T^{\lambda(\nu)} \restriction c_1$. Since $T^{\lambda(\nu)}$ is tubby at c_1, there is a unique point x^*_ν in $T^{\lambda(\nu)}_{c_1}$ extending this branch. Clearly, $\langle x^*_\nu \mid \nu < \theta \rangle \in T^*_{c_1}$. We put each such sequence $\langle x^*_\nu \mid \nu < \theta \rangle$ into T^λ_1. That completes the definition of $T_{\alpha+1}$ in all cases.

Now suppose that $T^\lambda \restriction \alpha$ is defined, where $\lim(\alpha)$, and we wish to define T^λ_α.

For each triple (ν, y, x) with $\nu < \theta$, $y \in T^{\lambda(\nu)}_{c_\alpha}$, $x \in T^* \restriction \alpha$, and $x_\nu \leqq_{T^{\lambda(\nu)}} y$, pick some $s = s(\nu, y, x) \in T^*_\alpha$ such that $x \leqq^* s$, $s_\nu = y$, and $\{x' \in T^\lambda \restriction \alpha \mid x' \leqq^* s\}$ is an α-branch of $T^\lambda \restriction \alpha$, and put this into T^λ_α. (That such an s can always be found is proved in [3], p. 91.) The actual choice of s is irrelevant except when

(**) $c_\alpha = \alpha \in E$, and $S_\alpha \subseteq T^\lambda \restriction \alpha$, and for every (ν, y, x) there is an $x' \in S_\alpha$ with ν, y, x' related as ν, y, x are related, and $x \leqq^* x'$.

In this case we select $s(\nu, y, x)$ so that $x' \leqq^* s(\nu, y, x)$ for some $x' \in S_\alpha$.

That defines T^λ_α completely except in the case $\alpha \in (\Omega - E) \cap C_\lambda$. We then enlarge T^λ_α to include an extension of each α-branch of $T^\lambda \upharpoonright \alpha$ lying in N_α. Since each $T^{\lambda(\nu)}$ ($\nu < \theta$) is tubby at α, each such branch will have an extension in T^*_α, so there is no problem in doing this. That completes the definition.

We set $T^\lambda = \bigcup_{\alpha<\omega_1} T^\lambda \upharpoonright \alpha$. Clearly, T^λ is a normal tree of height ω_1. The same argument as in [3], pp. 92/93 now shows that T^λ is Souslin. (Condition (**) above was designed to ensure that T^λ would have no uncountable antichain. The only difference between the proof in [3] and the one needed here is that in [3] we have a $\diamondsuit$-sequence, whereas here we have a $\diamondsuit(E)$-sequence. But this makes only a minor difference to the proof.) And it is clear that T^λ is tubby at every α in $(\Omega - E) \cap C_\lambda$.

Finally, suppose $\langle \mathbf{B}_\nu \mid \nu < \lambda \rangle$ is defined where λ is a limit ordinal cofinal with ω_1. Set $\mathbf{B}_\lambda = \bigcup_{\nu<\lambda} \mathbf{B}_\nu$. We find a Souslinisation of $\mathbf{B}_\lambda$. This will prove, in particular, that $\mathbf{B}_\lambda$ satisfies the c.c.c., and hence is complete. Our proof will also show that $\mathbf{B}_\nu$ is a nice subalgebra of $\mathbf{B}_\lambda$ for all $\nu < \lambda$.

Let $\langle \lambda(\nu) \mid \nu < \omega_1 \rangle$ enumerate A_λ. Define club sets $\mathbf{B}_\nu \subseteq \omega_1$, $\nu < \omega_1$, as follows:

$$B_0 = \Omega;$$

$$B_{\nu+1} = B_\nu \cap \left[\bigcap_{\tau<\nu} C_{\lambda(\tau),\lambda(\nu)} \right] \cap \left[\bigcap_{\tau\leqq\nu} C_{\lambda(\tau),\lambda(\nu+1)} \right];$$

$$B_\gamma = \bigcap_{\nu<\gamma} \mathbf{B}_\nu, \text{ if } \lim(\gamma).$$

Since $B_0 \supseteq B_1 \supseteq \cdots \supseteq B_\nu \supseteq \cdots (\nu < \omega_1)$, we can pick a strictly increasing, continuous sequence $\langle \beta_\nu \mid \nu < \omega_1 \rangle$ so that $\beta_0 = 0$ and $\beta_\nu \in B_\nu$ for all $\nu < \omega_1$. Then, for all $\gamma < \omega_1$,

$$\nu < \gamma \to T^{\lambda(\nu)} \upharpoonright \beta_\gamma = T^{\lambda(\nu)} \cap V_{\beta_\gamma};$$

$$\nu \leqq \tau < \gamma \to T^{\lambda(\nu)}_{\beta_\gamma} = h_{\lambda(\tau),\lambda(\nu)}[T^{\lambda(\tau)}_{\beta_\gamma}];$$

$$\nu \leqq \gamma \to T^{\lambda(\nu)}_{\beta_{\gamma+1}} = h_{\lambda(\gamma+1),\lambda(\nu)}[T^{\lambda(\gamma+1)}_{\beta_{\gamma+1}}].$$

By recursion, define sets $T_\nu \subseteq \mathbf{B}_\lambda$, $\nu < \omega_1$, as follows:

$$T_0 = \{\mathbf{1}\};$$

$$T_{\nu+1} = T^{\lambda(\nu+1)}_{\beta_{\nu+1}};$$

$$T_\alpha = \left\{ \bigwedge_{\nu<\alpha}^{\mathbf{B}_{\lambda(\alpha)}} x_\nu \;\middle|\; \bigwedge_{\nu<\alpha}^{\mathbf{B}_{\lambda(\alpha)}} x_\nu > \mathbf{0} \ \&\ \nu < \alpha \to x_\nu \in T^{\lambda(\nu)}_{\beta_\alpha} \ \&\right.$$

$$\left. \&\ \nu \leqq \tau < \alpha \to x_\nu = h_{\lambda(\tau),\lambda(\nu)}(x_\tau) \right\}, \text{ if } \lim(\alpha).$$

Set $T = \bigcup_{\nu<\omega_1} T_\nu$. T is clearly a normal tree of height ω_1 which is dense in $\mathbf{B}_\lambda$. The same argument as in [3], pp. 94–96 shows that T is Souslin. (Condition (*) in the definition of successor levels of $T^{\lambda(\nu)}$, $\nu < \omega_1$, was designed to ensure this, as the reader familiar with [3] will have already realised.) The argument on p. 96 of [3] also shows that if we set

$$C_{\lambda,\nu} = \{\alpha \mid \beta_\alpha = \alpha\}, \quad \text{for} \quad \nu < \lambda,$$

then $h_{\lambda\nu}[T^\lambda_\alpha] = T^\nu_\alpha$ for all $\alpha \in C_{\lambda\nu}$, so $\mathbf{B}_\nu$ is a nice subalgebra of $\mathbf{B}_\lambda$, all $\nu < \lambda$.

We must define C_λ and show that T^λ is tubby at every member of $C_\lambda \cap (\Omega - E)$.

Let $K = C_{\lambda,1} \cap \{\alpha \mid \alpha \in \bigcap_{\beta<\alpha} C_{\lambda(\beta)}\} \cap \{\alpha \mid \lambda(\alpha)$ is a limit point of A_λ and $\alpha = \text{otp}(A_{\lambda(\alpha)})\}$. K is a club set.

Let H be a club such that for all $\alpha \in H$,

$$\langle T^{\lambda(\nu)} \restriction \alpha \mid \nu < \alpha \rangle, \langle h_{\lambda(\tau),\lambda(\nu)} \restriction (T^{\lambda(\tau)} \restriction \alpha) \mid \nu < \tau < \alpha \rangle,$$

$$\langle \beta_\nu \mid \nu < \alpha \rangle \in N_\alpha.$$

Let $C_\lambda = K \cap H$. Let $\alpha \in (\Omega - E) \cap C_\lambda$. We show that T^λ is tubby at α. Notice that $A_{\lambda(\alpha)} = A_\lambda \cap \lambda(\alpha)$, and that $\text{otp}(A_{\lambda(\alpha)}) = \alpha$.

Recall the definition of $T^{\lambda(\alpha)}$. $T^{\lambda(\alpha)}$ consists of certain sequences $x = \langle x_\nu \mid \nu < \alpha \rangle$ such that for some $\gamma \in C$,

$$(\nu < \gamma \rightarrow x_\nu \in T^{\lambda(\nu)}_\gamma) \;\&\; (\nu < \tau < \gamma \rightarrow x_\nu = h_{\lambda(\tau),\lambda(\nu)}(x_\tau)),$$

where $C = (\bigcap_{\nu<\tau<\alpha} [C_{\lambda(\nu),\lambda(\tau)} - \alpha]) \cup \{0\}$.

Let $\langle c_\nu \mid \nu < \omega_1 \rangle$ be the canonical enumeration of C. Since $\alpha = \beta_\alpha \in B_\alpha = \bigcap_{\nu<\tau<\alpha} C_{\lambda(\nu)\lambda(\tau)}$, $\alpha \in C$. So as $0 \in C$ and $C \cap \alpha = \{0\}$, we must have $\alpha = c_1$.

Now let $\langle x_\nu \mid \nu < \alpha \rangle$ be an α-branch of $T^\lambda \restriction \alpha$, lying in N_α. Then $\nu < \alpha \rightarrow x_{\nu+1} \in T^{\lambda(\nu+1)}_{\beta_{\nu+1}}$ and $\nu < \tau < \alpha \rightarrow x_{\nu+1} \leqq_{T^{\lambda(\nu+1)}} h_{\lambda(\tau+1),\lambda(\nu+1)}(x_{\tau+1})$. But $\alpha \in \bigcap_{\nu<\alpha} C_{\lambda(\nu)}$, so by construction of $T^{\lambda(\nu)}_\alpha = T^{\lambda(\nu)}_{c_1}$, $\langle x_{\nu+1} \mid \nu < \alpha \rangle$ has an extension, x, in $T^{\lambda(\nu)}_\alpha$. Clearly, x extends $\langle x_\nu \mid \nu < \alpha \rangle$ in T^λ_α. The proof is complete.

That completes the construction of the sequence $\langle \mathbf{B}_\nu \mid \nu < \omega_2 \rangle$. Let $\mathbf{B} = \bigcup_{\nu<\omega_1} \mathbf{B}_\nu$. Since $\mathbf{B}$ will satisfy the c.c.c., it is a complete boolean algebra. Moreover, $\mathbf{B}$ will be (ω, ∞)-distributive, since each $\mathbf{B}_\nu$, $\nu < \omega_2$, is. We complete our proof of the main theorem by proving the following lemma.

4.1 Lemma.

(i) $\|\Diamond(\Omega - E)\|^{\mathbf{B}} = \mathbf{1}$,

(ii) $\|\text{SAD}(E)\|^{\mathbf{B}} = \mathbf{1}$.

Proof. Notice first that by 3.2, $\|\check{E}$ and $(\Omega - E)^\vee$ are stationary$\|^{\mathbf{B}} = \mathbf{1}$.

(i) It suffices to show that for all $\nu < \omega_2$, $\|\diamondsuit(\Omega - E)\|^{\mathbf{B}_\nu} = \mathbf{1}$. But T^ν is a Souslinisation of $\mathbf{B}_\nu$ which is tubby at every element of $C_\nu \cap (\Omega - E)$, so this follows from 3.3.

(ii) Suppose $\|\mathrm{SAD}(E)\|^{\mathbf{B}} < \mathbf{1}$. Thus for some $D \in V$, D a filter array on E, $\|\mathrm{SAD}(\check{E})$ holds on $\check{D}\,\|^{\mathbf{B}} < \mathbf{1}$. By the maximum principle, we can find a $\overset{0}{T} \in V^{(\mathbf{B})}$ such that

(a) $\|\overset{0}{T}$ is a function tree appropriate for $\check{D}\,\|^{\mathbf{B}} = \mathbf{1}$;

(b) $\|\overset{0}{T}$ has an $\check{\omega}_1$-branch$\|^{\mathbf{B}} < \mathbf{1}$.

Pick $\delta < \omega_2$ with $\overset{0}{T} \in V^{(\mathbf{B}_\delta)}$. If $D = \{D_{\alpha,f} \mid \alpha \in E \ \&\ f \in \omega^{\underline{\alpha}}\}$, well-order $D_{\alpha,f}$ by $\langle A_\xi^{\alpha,f} \mid \xi < \omega_2\rangle$. Let $E_{\alpha,f}$ be the set of all $\xi < \omega_2$ such that

$$\|\text{if } \check{f} \in \overset{0}{T}, \text{ and if } \check{f} \subseteq h \in (A_\xi^{\alpha,f})^\vee, \text{ and if } (\forall \eta < \check{\alpha})(h \restriction \eta \in \overset{0}{T}),$$

$$\text{then } h \in \overset{0}{T}, \text{ and } \check{\xi} \text{ is least for which this occurs}\|^{\mathbf{B}_\delta} > \mathbf{0}.$$

Since $\mathbf{B}_\delta$ satisfies the c.c.c., $E_{\alpha,f}$ is countable. Set

$$A^{\alpha,f} = \bigcap_{\xi \in E_{\alpha,f}} A_\xi^{\alpha,f}.$$

Since $D_{\alpha,f}$ is countably complete, $A^{\alpha,f} \in D_{\alpha,f}$. Let $D'_{\alpha,f}$ be the principal filter generated by $A^{\alpha,f}$. Let $D' = \{D'_{\alpha,f} \mid \alpha \in E \ \&\ f \in \omega^{\underline{\alpha}}\}$. D' is a principal filter array. For some $\beta < \omega_2$, $D' = D_\beta$. For some $\gamma < \omega_2$, $\overset{0}{T} = \overset{0}{T}{}^{\delta,\gamma}$. Let $\nu = \phi(\delta, \beta, \gamma)$. By construction of $\mathbf{B}_{\nu+1}$, $\|\overset{0}{T}$ has an $\check{\omega}_1$-branch$\|^{\mathbf{B}_{\nu+1}} = \mathbf{1}$. This contradicts (b) above, and so completes the proof. (The other clauses of SAD(E) are, of course, immediate.) □

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